ANTENNA LABORATORY

Technical Report No. 13

IMPEDANCE OF FERRITE LOOP ANTENNAS

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V. H. Rumsey

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15 October 1956

Contract No. AF33(616)-3220
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ABSTRACT

A variational formula for the imput impedance of a loop antenna with a ferrite core has been derived in terms of an assumed volume distribution of magnetization in the core. It takes the form

$$Z - Z_o = \frac{\left[\iint_S \frac{\underline{J} \cdot \underline{E}_x}{I} \, dS\right]^2}{\iiint_{core} \underline{K} \cdot \left(\frac{\underline{K}}{j\omega(\mu - \mu_o)} - H_k\right) dV}$$

where Z = impedance of the antenna

Z_o = impedance when the ferrite is removed

$\underline{J}$ = surface density of electric current on the winding, assumed perfectly conducting

$\underline{K}$ = assumed volume density of magnetic dipoles

μ and μ_o represent the permeability of the ferrite and free space

$\underline{E}_k$ and $\underline{H}_k$ represent the free space field due to $\underline{K}$.

Effects of the winding arrangement, core losses and shape of core are contained in the formula. It reduces to a simple expression when applied to an electrically small antenna with ellipsoidal core.

CONTENTS

LIST OF SYMBOLS

A	area
D	demagnetization factor
E	electric field intensity
E_f	E generated by ferrite magnetic dipoles in free space
E_t	component of E tangent to the path of the wire
E_w	E generated by wire currents in free space
H	magnetic field intensity
H_c	component of H in the core which is normal to plane of loop
H_f	H generated by ferrite magnetic dipoles in free space
H_o	magnetic field of plane wave incident on loop
H_w	magnetic field generated by wire currents in free space
I	electric current
J	surface density of electric current
K	density of induced magnetic dipoles
K_a	assumed (or approximate) distribution of K
N	unit vector normal to area
O	to the order of
P	refers to point at source of field
Q	refers to point of observation
R	resistance
R_R	radiation resistance
S	general surface
V	volume
$\mho$	input voltage
Z	input impedance of antenna
Z_o	Z of antenna in free space
Z_{oe}	Z_o for ellipsoidal antenna winding
Z_{oL}	Z_o for single turn loop
a	length of ellipsoid semi-axis (2a is length of core measured along the length of the winding)
a_p	symbolic representation for the source consisting of the assumed (or approximate) distribution of magnetic dipoles in the ferrite
b	ellipsoid semi-axis, or radius of circular loop around the core

c	ellipsoid semi-axis
f	symbolic representation of source consisting of the volume distribution of induced magnetic dipoles
j	$\sqrt{-1}$
k	relative permeability, $\frac{\mu}{\mu_o}$ (in general, $k = k' - j\,k''$)
k'	real part of permeability
k''	imaginary part of permeability
l	length along the wire path
n	number of turns on the winding
r	distance from P to Q
s	spacing between turns measured along the axis $(\frac{2a}{n})$
u	symbolic representation of source, consisting of assumed distribution of induced magnetic dipoles in ferrite
w	symbolic representation of the source consisting of the electric currents in the wire
$\hat{z}$	unit vector in z direction (direction of the a-axis)
β	free space propagation constant
μ	permeability
μ_o	μ for free space
λ	free space wavelength
ω	circular frequency

$\langle xy \rangle$ reaction between the sources x and y. In general

$\langle xy \rangle \equiv \iiint (E_x J_y H_x K_y)\, dV.$

$\langle xy \rangle = \langle yx \rangle$ by the reciprocity theorem.

1. INTRODUCTION

The recent development of ferrites having comparatively low loss and high permeability has opened the possibility of improving the performance of small loop antennas by the introduction of a ferrite core. In receiving antennas, which is the application we have in mind, the improvement, if any, would appear as an increase in the signal-to-noise ratio for an antenna of a given size. Since the signal-to-noise ratio is implicitly determined by the input impedance, the essence of the problem is to calculate the input impedance of a length of wire wound around a ferrite core, for various core shapes and winding arrangements.

The typical antenna under discussion has a diameter of a few centimeters at 10 mc. It follows that the contribution to the input resistance due to ohmic losses in the winding can be treated separately from the contributions due to core losses and radiation. The resistance due to imperfect conductivity of the winding is therefore considered in a separate study.

It is assumed that the ferrite is uniform, linear, and isotropic. For the sake of simplicity it is also assumed that the permittivity of the core is that of free space, an assumption which, though incorrect, has only a slight effect on the results.

It is apparent at the outset that it would be impractical to attempt an exact solution based on Maxwell's equations. We therefore use the reaction concept[1] to derive approximate formulas based on certain assumed field distributions. The method to be discussed is based on an assumed distribution of magnetic field in the core. Since the antenna is electrically small, a good approximation is obtained by using the magnetostatic solution as the assumed distribution. This method is therefore practically restricted to ellipsoidal shapes because it is only for such shapes that the magnetostatic solution (as represented by the "demagnetization factor") is available.

The formulas used have the usual stationary properties. At times this fact is hardly relevant since the assumed distributions are sometimes nowhere near correct. What is interesting is that, despite a gross error in the assumed distribution, the radiation resistance is given correctly by the formulas. Indeed, it should be noted that the radiation resistance R_R, can be calculated with good accuracy by using

a much simpler approach than that given in the following sections for the calculation of the impedance (i.e., reactance and resistance due to radiation and core losses). In outline, this simple method proceeds as follows: The antenna is represented as a magnetic dipole. The reciprocity theorem is used to find the dipole moment in terms of the loop current and the distribution of magnetic field, $\underline{H}_c$, set up in the core by an incident plane wave. The power radiated by the dipole (which can be easily derived from a distant field calculation) is then set equal to the input power $\frac{1}{2}|I|^2 R_R$, thus giving a formula for R_R in terms of the distribution, $\underline{H}_c$. The result for a single turn loop is

$$R_R = \frac{31{,}171}{\lambda^4} \left| \iint_A \frac{k\, H_c}{H_o}\, dA \right|^2 \tag{1}$$

where

λ = free space wavelength

H_o = incident magnetic field (which is taken normal to plane of loop)

H_c = component normal to plane of loop of magnetic field in core due to incident plane wave

A = area of loop

k = relative permeability, μ/μ_o, of core (k in general is complex), and the usual $e^{j\omega t}$ time convention is implied. The quantity H_c/H_o is practically the same as for the magnetostatic case (because the antenna is electrically small). When the shape of the core is ellipsoidal, the static value of H_c/H_o is a constant. Then Eq. 1 reduces to

$$R_R = \frac{31{,}171}{\lambda^4} \left| \frac{k\, n\, A}{1 + \mathcal{D}\,(k - 1)} \right|^2 \tag{2}$$

where

$\mathcal{D}$ = demagnetization factor (from static solution)

n = number of turns.

Thus, a ferrite core which completely fills the loop increases the radiation resistance by a factor of $|k/[1 + \mathcal{D}\,(k - 1)]|^2$. For many ferrites, this factor can easily be made of the order of a thousand. Even so, for typical antenna sizes, the radiation resistance is only of the order of 10^{-6} ohms at 10 mc/sec.

The formulas for the impedance are obtained from the basic relation

(which is derived in Appendix I)

$$I^2 Z = -\iint_S \underline{J} \cdot \underline{E} \, dS$$

wherein

I = input current at the antenna terminals

Z = input impedance.

J is the surface density of electric current on the surface, S, of the metal part of the antenna, and $\underline{E}$ is the electric field that would be generated by $\underline{J}$ if said metal part were removed, leaving free space in its place. (It is assumed that the "metal" is a perfect conductor and that I, Z, $\underline{J}$, and $\underline{E}$ are complex according to the $e^{j\omega t}$ time convention.) Equation 3 is exact provided the terminals are an infinitesimal distance apart and located at opposite sides of a gap in the metal structure. If the antenna consists of a metal loop of wire, the surface, S, is the surface of the wire, $\underline{J}$ is the *surface* density of current on the wire and thus, ignoring the insignificant contribution from the gap,

$$\iint_S \underline{J} \cdot \underline{E} \, dS = \oint I(l) \, E_t dl$$

where $I(l)$ is the *line* density of current as a function of distance, l, along the wire and E_t is the component of $\underline{E}$ in the direction of the differential increment, dl, (note that $\underline{E}$ is the field generated by $I(l)$ *with the wire removed*). For the "electrically small" case, $I(l)$ is practically constant and equal to I, so that Eq. 3 becomes

$$IZ = -\oint E_t \, dl. \tag{4}$$

If the wire consists of a coil of several turns, Eq. 4 still applies, provided $\oint$ is understood as following the wire from one input terminal to the other. If the turns are closely spaced, E_t is practically the same for each turn and Eq. 2 reduces to

$$IZ = -n \oint E_t \, dl \tag{5}$$

where n is the number of turns. Equations 4 and 5 are well-known in connection with conventional loop antennas.

2. FORMULAS BASED ON AN ASSUMED DISTRIBUTION IN THE CORE

The field generated by $\underline{J}$ in the presence of the core is the same as the superposition of the fields generated in the absence of the core by $\underline{J}$ and the volume density of magnetic dipoles (or currents), $j\omega(\mu-\mu_o)\underline{H}$, where $\underline{H}$ is the magnetic field generated by $\underline{J}$ in the presence of the core and μ and μ_o are the permeabilities of the core and free space, respectively. To distinguish these two sources, let w denote the wire electric currents $\underline{J}_w = \underline{J}$ (surface density) and f denote the ferrite magnetic dipoles $\underline{K}_f = j\omega(\mu-\mu_o)\underline{H}$ (volume density). Let $\underline{E}_w$, $\underline{H}_w$, $\underline{E}_f$, $\underline{H}_f$ denote the corresponding fields in free space, and $\underline{E}$,$\underline{H}$ the field of w in the presence of the core.

Then

$$\underline{E} = \underline{E}_w + \underline{E}_f$$

and Eq. 3 can thus be written

$$I^2 Z = -\iint_S \underline{J}_w \cdot (\underline{E}_w + \underline{E}_f) dS \tag{6}$$

or using the notation of the reaction concept

$$I^2 Z = -\langle ww\rangle_{\text{no core}} - \langle wf\rangle_{\text{no core}} \tag{7}$$

and similarly Eq. 3 can be written

$$I^2 Z = -\langle ww\rangle_{\text{with core}}. \tag{8}$$

On the assumption of uniform current in the winding, $\underline{J}$ is practically the same with or without the core for a given input current.

$$\therefore -\langle ww\rangle_{\text{no core}} = I^2 Z_o \tag{9}$$

where Z_o is the input impedance with no core. Therefore Eq. 7 takes the form

$$I^2 (Z - Z_o) = -\langle wf\rangle_{\text{no core}}. \tag{10}$$

Assuming that Z_0 is known, the problem therefore reduces to finding an approximation for <wf>. Thus, this approach consists essentially of finding the extra effect of the core, assuming that the impedance with no core is known.

To find an approximation from <wf> we start with an assumed volume distribution, denoted by $\underline{K}_a$, which we are going to use as an approximation to $\underline{K}_f$. By using the exact relation $\underline{K}_f = j\omega(\mu-\mu_o)(\underline{H}_w + \underline{H}_f)$ (which follows from the definitions of the f and w fields) a variety of different formulas can be obtained for $\langle w_f \rangle$ (see Appendix II). If, in these formulas, we replace the $\underline{f}$ field by its approximation, the a_p field, the different formulas no longer give the same result--they are all different approximations for $\langle w_f \rangle$. The optimum approximation is therefore obtained if we can adjust a_p to make all of these different approximations give the same result. If we rule out those forms which require an explicit knowledge of $\underline{H}_w$, the free space magnetic field of the loop--a complicated and practically unmanageable function in most cases--we are left with three different forms which can be made to give the same result by adjusting the level of the assumed $\underline{K}_a$. The equation so obtained is (see Appendix II)

$$\langle wa_p \rangle = - \langle a_p\, a_p \rangle - \iiint_{core} \frac{\underline{K}_a \cdot \underline{K}_a}{j\omega(\mu-\mu_o)}\, dV \tag{11}$$

or

$$\iint_S \underline{J}_w \cdot E_a\, dS = \iiint_{core} \underline{K}_a \cdot \left[\underline{H}_a - \frac{\underline{K}_a}{j\omega(\mu-\mu_o)}\right] dV. \tag{12}$$

Making use of this result and replacing <wf> by $\langle wa_p \rangle$ in Eq. 10, we obtain the following approximation for the input impedance:

$$Z - Z_0 = \frac{\left[\iiint_S \frac{\underline{J}_w \cdot \underline{E}_u\, dS}{I}\right]^2}{\iiint_{core} \underline{K}_u \cdot \left(\frac{\underline{K}_u}{j\omega(\mu-\mu_o)} - \underline{H}_u\right) dV}. \tag{13}$$

The u field is the assumed distribution--it differs from the a_p field by a constant. (The distinction between the u and a_p fields is that the level of the u field does not matter in Eq. 13 whereas the level of the a_p field, if we use it in place of the $\underline{f}$ field, does matter. (Equation 13 automatically ensures that a_p takes on the level determined

by Eq. 12.) For a loop consisting of a single turn

$$\iint_S \underline{J} \cdot \underline{E}\, dS = - I \oint E_t \, dl \qquad \text{(see Eq. 4)}$$

$$= - I \iint_A (\underline{\nabla} \times \underline{E})_N \, dA$$

where A is the area of the loop and N the direction normal to A so that the current is right-handed about N

$$= I \iint_A (j\omega\mu_o \underline{H} + \underline{K})_N \, dA \qquad \text{(Maxwell's equation).}$$

Thus the problem of having to find $\underline{E}_u$ can be avoided by writing

$$(1/I) \iint_S \underline{J}_w \cdot \underline{E}_u \, dS$$

in the form

$$\iint_A (j\omega\mu_o \underline{H}_u + \underline{K}_u)_N \, dA. \qquad (13A)$$

For a distributed winding, the same device can be used to a good approximation, by applying this relation to each turn, ignoring the fact that the beginning and end of a turn are not quite at the same point. Thus, for a closely spaced winding, uniformly distributed along the axis over the entire extent of the core,

$$(1/I) \iint_S \underline{J}_w \cdot \underline{E}_u \, dS$$

can be replaced by

$$\frac{n}{2a} \iiint_V (j\omega\mu_o \underline{H}_u + \underline{K}_u)_N \, dV \qquad (13B)$$

where n is the number of turns and 2a the length of the core measured along the axis of the winding.

3. FORMULAS IN TERMS OF THE DEMAGNETIZATION FACTOR

We note that in order to use the formula (Eq. 13) for Z, we must find the magnetic field $\underline{H}_u$ due to the assumed volume density of magnetic dipoles, K_u. [The electric field $\underline{E}_u$ is not required provided the winding of the loop allows us to use the transformations (13A) or (13B).] The aim of this section is to describe the calculation of $\underline{H}_u$ and its relation to the demagnetization factor.

The magnetic field, $\underline{H}_f$, generated in free space by a volume density, $\underline{K}$, of magnetic current (or dipoles) is given by

$$j\omega\mu_o \underline{H}_f (Q) = \nabla_Q \times \nabla_Q \times \iiint_V \underline{K}(P) \frac{e^{-j\beta r}}{4\pi r} dV \qquad (14)$$

where β is the free space propagation constant, r is the distance from P to Q, P is a point in the source, and Q is the point of observation. It will be noted from Eq. 13 that we need the value of $\underline{H}_u$ only at points within V. Thus the largest value of r involved is the maximum dimension of the core, which is small compared with the wavelength $2\pi/\beta$. Expansion of the exponential therefore gives a rapidly convergent series:

$$\frac{e^{-j\beta r}}{r} = \frac{1}{r} - j\beta - \frac{\beta^2 r}{2} + j\,\frac{\beta^3 r^2}{6} \cdots \qquad (15)$$

The first term gives the static solution (corresponding to $\beta = 0$). The second term contributes nothing. Thus if $\underline{K}$ is real, the first contribution to the real part of $\underline{H}_f$ comes from the fourth term, which happens to give a particularly simple integrand. The first order approximations to the real and imaginary parts of $\underline{H}_f$ can therefore be obtained fairly easily from Eq. 14 if the static solution is known. Thus we try to find some $\underline{K}$ for which the static solution is known and which is as close as possible to the correct $\underline{K}$. Obviously the best choice would be the solution for the given antenna when energized with DC current but unfortunately this is too complicated except for a few special cases. The only really simple static solution is that for an ellipsoid in a uniform magnetic field, which gives a uniform resultant field inside the ellipsoid. To utilize this solution we therefore assume a uniform $\underline{K}$ and restrict the core shape to an ellipsoid. Then the contribution to Eq. 14 from the "static" term can be written in the form, for points in V,

$$j\omega\mu_o \underline{H}_f = - \mathbb{D}\underline{K} \quad (\underline{K} \text{ is constant}) \qquad (16)$$

where $\mathcal{D}$ is the demagnetization factor, the value of which is known from the static solution:[2] it depends on the shape of the ellipsoid and is independent of μ (see also Figs. 1 and 2). Thus, taking $\underline{K}$ to be real,

$$\text{Im } \underline{H}_f = \frac{\mathcal{D}K}{\omega\mu_o} [1 + O(\beta L)^2] \tag{17}$$

and from the r^2 term in Eq. 15 one obtains

$$\text{Re } \underline{H}_f = \frac{\underline{K}\beta^3 V}{\omega\mu_o 6\pi}[1 + O(\beta L)^2] \tag{18}$$

where L is the largest dimension of the core. Substitution from Eqs. 17 and 18 in Eq. 13 gives

$$Z - Z_o \approx \frac{A^2\omega\mu_o(k - 1)\ (1 - \mathcal{D})\ (j[1 - \mathcal{D}] + \frac{\beta^3 V}{6\pi}\ [1 + \frac{k}{1 + \mathcal{D}(k - 1)}])}{V\ (1 + \mathcal{D}\ [k - 1])} \tag{19}$$

for an ellipsoidal core with a single loop located in any plane of symmetry, where

$k = \mu/\mu_o$ = relative permeability of the core
A = area of the loop
V = volume of the core.

[It should be pointed out that if k is real the real part of Eq. 19, which represents the increase in radiation resistance due to the ferrite, agrees with the value obtained from the far field method of calculation (see Eq. 1)].

Similarly, for a winding of many turns *uniformly spaced along an axis* of the ellipsoid

$$Z - Z_o = \frac{V\omega\mu_o n^2(k - 1)(1 - \mathcal{D})}{(2a)^2[1 + \mathcal{D}(k - 1)]}(j[1 - \mathcal{D}] + \frac{\beta^3 V}{6\pi}\ [1 + \frac{k}{1 + \mathcal{D}(k - 1)}]) \tag{20}$$

where n = number of turns
2a = length of said axis.

This result serves as a check on the imaginary part of $(Z - Z_o)$ because when $n \to \infty$ the assumed field becomes correct in the static approximation. Therefore if k is real the imaginary part of $(Z - Z_o)$ should agree to $O(\beta l)^2$ with the (exact) static calculation of inductance, which it does

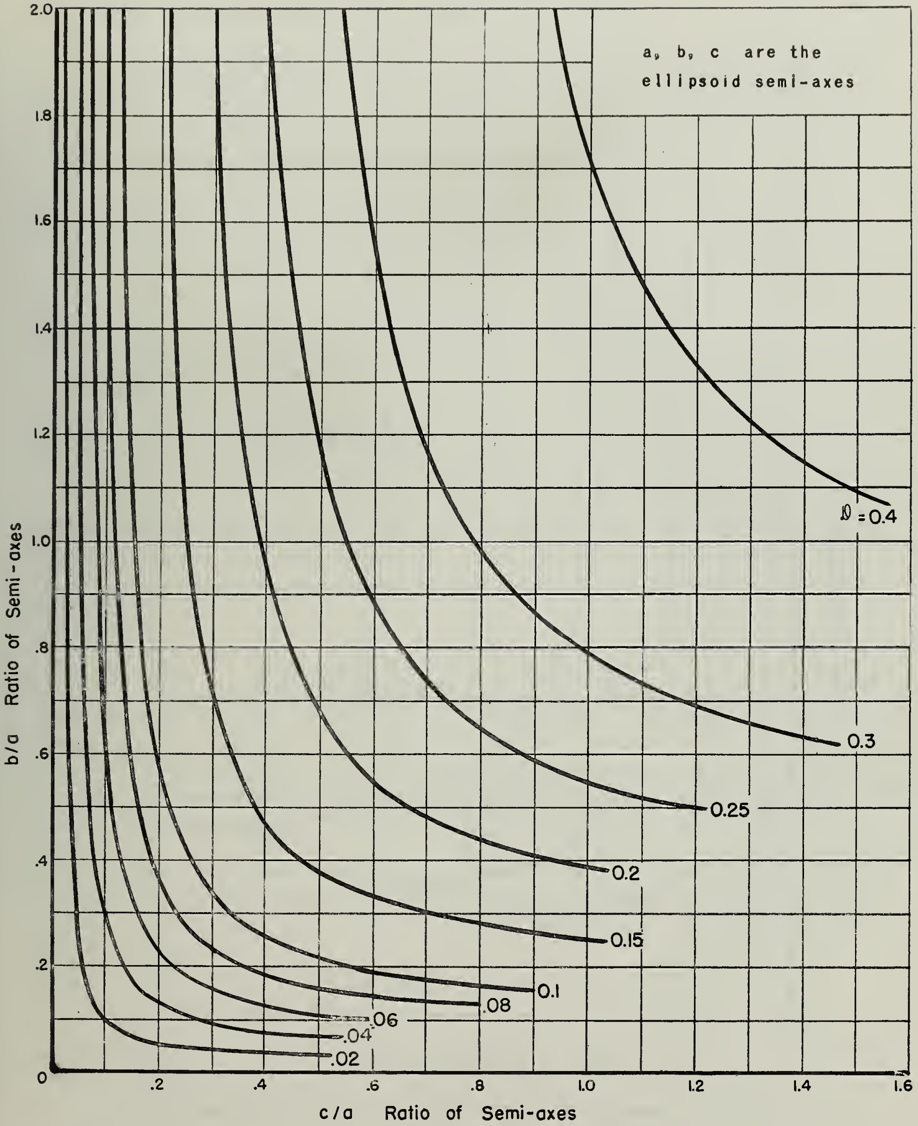

Figure 1 Demagnetization Factors of Ellipsoids along the a Semi-Axis

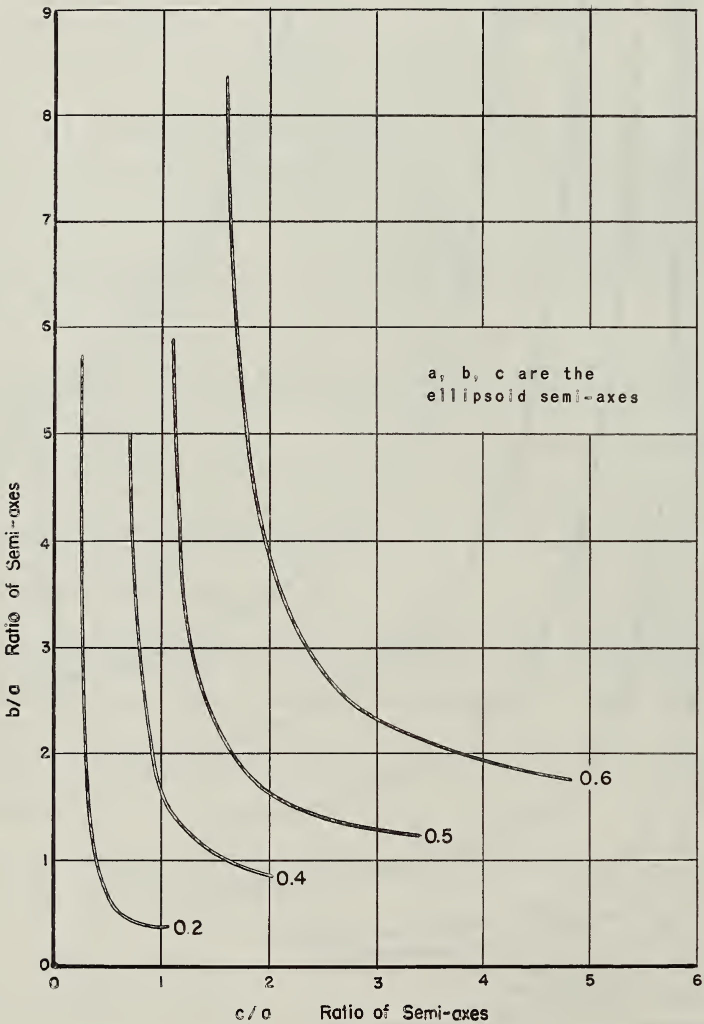

Figure 2 Demagnetization Factors of Ellipsoids along the <u>a</u> Semi-Axis

(see Eq. 22, Appendix IV, and also Reference 4.).

It should be noted also that k, in Eq. 19 or 20, can be either real or complex. Thus, since it has been found possible to represent the behavior of ferrites in terms of a complex permeability,[3] these equations give the effect on input impedance of core losses.

To find the actual value of the impedance from Eqs. 19 or 20, the impedance of the coil in free space, Z_o, must be known. The value to be used in Eq. 19 (the free space impedance of a single turn loop) is

$$Z_{oL} = j\omega\mu_o b\ [ln\frac{16b}{d} - 2] + 31,171 \left(\frac{\pi b^2}{\lambda^2}\right)^2 \tag{21}$$

in which

b = the radius of the loop

and

d = the diameter of the wire.

(The greatest accuracy is not to be expected for the reactive part of the impedance found from Eqs. 19 and 21 since the assumed distribution is a poor approximation to the actual.) The value of Z_o to be used in Eq. 20 is

$$Z_{oe} = \frac{j\omega\mu_o n^2(1 - \mathcal{D})V}{(2a)^2} + \frac{31,171}{\lambda^4}\left(\frac{nV}{2a}\right)^2 \tag{22}$$

in which

a = the semi-axis of the ellipsoid which is perpendicular to the planes of the turns.

As was pointed out earlier, the quantity, $\mathcal{D}$, to be used in the formulas can be found in the literature.[2] It can also be found from the equation

$$\mathcal{D}_j = \frac{a_1a_2a_3}{2}\int_0^\infty \frac{ds}{(s + a_j)^2\sqrt{(s + a_1^2)(s + a_2^2) + (s + a_3^2)}}, \quad j = 1, 2, 3$$

in which a_1, a_2, a_3, are the semi-major axes of the ellipsoid and $\mathcal{D}_j$ is the demagnetization factor along the j^{th} semi-axis. For convenience, graphs which show the effect of the shape on the demagnetization factor are presented (Figs. 1 and 2). The quantity $\mathcal{D}$ which is plotted in these figures is really $\mathcal{D}_1$, the value along one of the axes--namely, it is the value

along the axis which is perpendicular to the planes of the turns of the winding. The curves are symmetric about the 45° line.

It is interesting to consider the ratio of the radiation resistance to the resistance introduced by core losses for these ellipsoidal antennas. It follows from Eqs. 20 and 22 that, if $k = k' - jk''$,

$$\frac{R_{rad}}{R_{core\ loss}} \simeq \frac{\frac{4}{3}\pi k'^{2} \frac{V}{\lambda^{3}}}{k'' (1 - \mathcal{D})^{2}} ,$$

assuming that $k'' << k'$ (which is almost always the case). The variation of this ratio with $\mathcal{D}$, assuming constant value for the other parameters (especially antenna volume) is shown in Fig. 3. In addition we note the following: The ratio is:

1. directly proportional to the antenna volume, given a constant $\mathcal{D}$ and a particular ferrite.
2. directly porportional to the real part of the permeability, given the antenna volume and shape and a specified ferrite loss tangent.
3. directly proportional to the inverse of the loss tangent, given permeability, shape, and volume.

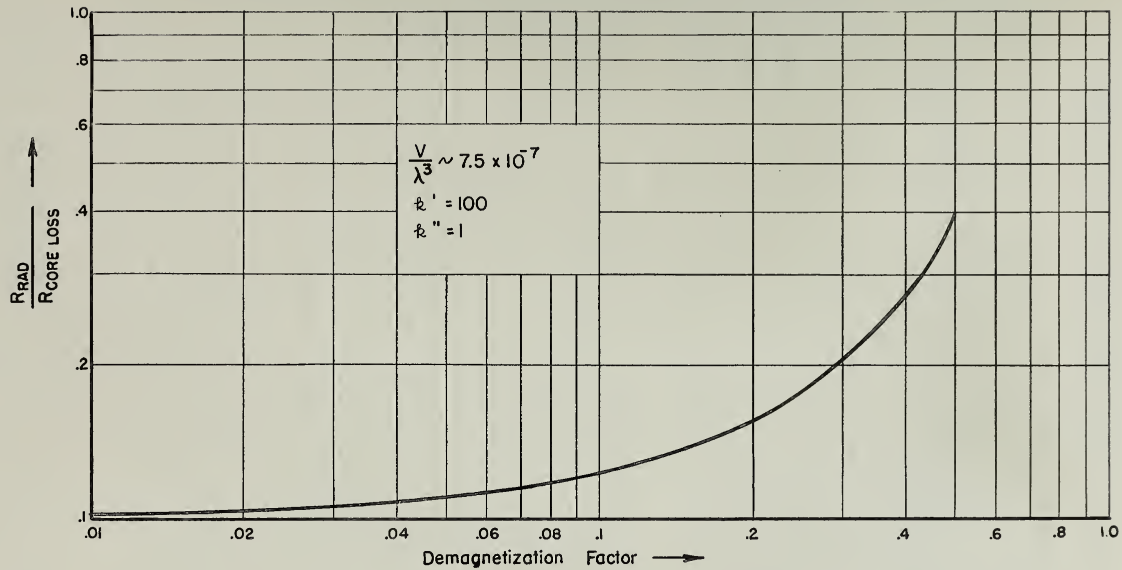

Figure 3 Effect of the Demagnetization Factor on the Ratio of the Resistive Components of Ellipsoidal Antennas. The Antenna Volume Is Assumed Constant.

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APPENDIX I

DERIVATION OF THE FORMULA $I^2 Z = -\iint_S \underline{J} \cdot \underline{E}\, dS$

The formula $I^2 Z = -\iint_S \underline{J} \cdot \underline{E}\, dS$ has been used for decades (in special forms) but there is considerable confusion about it in the literature, particularly in connection with an alternative formula which uses the idea of complex power, i.e., $|I|^2 Z = \iint_S \underline{J}^* \cdot \underline{E}\, dS$ where the star denotes complex conjugate. Since the formulas have always been used for *approximate* calculations based on an assumed *real* $\underline{J}$, the difference, if any, between the two formulas has never been uncovered. Since the first formula is valid *in general* for any perfectly conducting structure, as we shall shortly prove, in general $\underline{J}$ is not real and the two formulas are therefore distinctly different. There is a possibility that they might give the same Z but this appears to be very difficult to decide. Since we can demonstrate the correctness of the first formula, but not of the second, it is the first which we use.

The essence of the problem is to calculate the circuit theory parameter Z by using field theory. The key to the problem is therefore the connection between circuit theory and field theory. To establish the appropriate connection suppose that the antenna is energized by a constant voltage generator. Then we have to find the field theory source which is equivalent to a constant voltage generator. To do this we note that since the problem implies the existence of the circuit parameter Z, the terminals of the antenna must take the form of two points, A and B, an infinitesimal distance, l, apart, at opposite sides of a gap in the metal structure of the antenna, as illustrated in Fig. 4. Since it will turn out that we have to assume perfectly conducting metal in order to justify the formula, let us make that assumption now in order to define the problem more clearly. Now consider a uniform surface density $\underline{K}$ volts per meter of magnetic current flowing around a perfectly conducting cylinder, henceforth called the "plug", which is placed between the terminals as in Fig. 4. Since tangential E vanishes at the surface of the plug and is discontinuous at the magnetic current sheet by the amount $\underline{K}$, it follows that

$$\underline{E} = \underline{n} \times \underline{K}$$

just outside of the magnetic current sheet, $\underline{n}$ being a unit vector in the direction of the outward normal, as shown in Fig. 4. Since l is infinitesimal we can relate $\underline{E}$ to the circuit concept of input voltage, $\mathcal{V}$, by

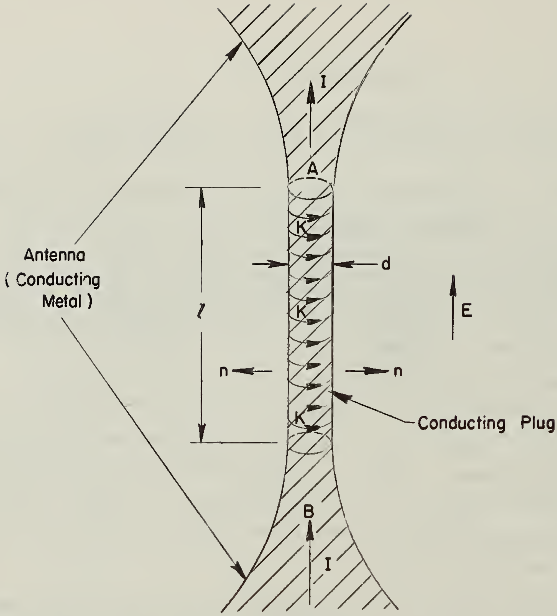

Figure 4 Antenna Energized by Magnetic Current Source (Equivalent to a Constant Voltage Generator)

the equation

$$\mho = \mho_A - \mho_B = - \int_B^A \underline{E} \cdot \underline{dl} = - El = - Kl,$$

the directions of E and K being as shown in Fig. 4. Thus if $\underline{K}$ is fixed, $\mho$ is fixed: the field theory equivalent of a constant voltage generator $\mho$ is a uniform solenoid of magnetic current of surface density $\mho/l$, completely filled with a plug of perfect electric conductor, of length l and diameter d, where $d/l \to o$ and $l \to o$.

Note that the addition of the plug is not trivial for, according to circuit theory, when a constant voltage generator is switched off, the internal impedance between the terminals is a short circuit, and according to field theory, when $\underline{K}$ is switched off, the terminals remain connected by the plug, which is a short circuit. Plainly, the internal impedance would not be a short circuit if the plug were absent.

The problem of a constant voltage generator connected to the antenna can thus be defined in terms of field theory as the problem of the source represented by $\underline{K}$ in the presence of the short-circuited antenna. Let S denote the entire surface of the perfect conductor which is formed by the

shorted antenna, i.e., S is the combination of the metal surface of the antenna and the surface of the plug: (S is a single closed surface if the antenna consists of two metal parts between which the terminals are situated). Let h denote the distribution of magnetic current, $\underline{K}$, and c denote the electric current distribution which is induced on S by h. Thus c denotes the combination of the antenna current distribution and the distribution of the current on the plug; it consists of a surface density $\underline{J}$ which is continuous at every point on S. (Note that the antenna current is not continuous everywhere because it stops at the input terminals). Now it follows from Maxwell's equations that the field due to h in the presence of the shorted antenna is identical to the superposition of the fields due to h and c in the absence of the shorted antenna. Symbolically, this can be expressed in the form

$$\underline{E}_{h1} = \underline{E}_{h0} + \underline{E}_{c0} \qquad \text{(i)}$$

$$\underline{H}_{h1} = \underline{H}_{h0} + \underline{H}_{c0} \qquad \text{(ii)}$$

where subscripts c and h denote the source of the field in question and the subscripts 0 and 1 represent the environment which exists when the antenna conductor is removed and when it is in place, respectively. (If the antenna consists entirely of the metal parts involved in the definition of S, then 0 represents free space).

We now bring in the reciprocity theorem by multiplying (*ii*) throughout by $\underline{K}$, the surface density of magnetic current which comprises the source h. Note that

$$\iint_{\text{over } \underline{K}} \underline{H}_{h1} \cdot \underline{K}\, dS = \iint_{\text{over } \underline{K}} (\underline{H}_{h0} \cdot \underline{K} + \underline{H}_{c0} \cdot \underline{K})\, dS \qquad \text{(iii)}$$

or, to state this result more precisely,

$$\iint \underline{H}_{h1} \cdot \underline{dK} = \iint \underline{H}_{h0} \cdot \underline{dK} + \iint \underline{H}_{c0} \cdot \underline{dK}. \qquad \text{(iv)}$$

Now since $d/l \to 0$ and $l \to 0$, $\underline{H}_{h1}$ and $\underline{H}_{h0}$ are parallel to $\underline{K}$ at points occupied by $\underline{K}$ and since K is uniform ($K = -\mho/l$),

$$\iint \underline{H}_h \cdot \underline{dK} \to -\mho \oint \underline{H}_h\, dS$$

where $\oint dS$ represents integration around a typical loop of the magnetic current solenoid. Thus, from Ampère's law

$$\iint \underline{H}_{h1} \cdot \underline{dK} \to -\mho I$$

where I = input current as in Fig. 4, and

$$\iint \underline{H}_{h0} \cdot \underline{dK} \to 0$$

since $\underline{H}_{h0}$ tends to the static theory limit of zero, as $l \to 0$ and $d/l \to 0$. Also from the reciprocity theorem

$$\iint \underline{H}_{c0} \cdot \underline{dK} = -\iint_{\text{over } c} \underline{E}_{h0} \cdot \underline{dJ}$$

where $\underline{J}$ is the surface density of electric current which comprises the source c. Substitution of these results in (iii) gives

$$\mho I = \iint_{\text{over } s} \underline{E}_{h0} \cdot \underline{dJ}. \qquad \text{(v)}$$

Finally we bring in the boundary condition that the tangential component of $\underline{E}_{h1}$ vanishes at S. Therefore, from (i),

$$\underline{N} \times \underline{E}_{h0} = -\underline{N} \times \underline{E}_{c0} \text{ on S}$$

where $\underline{N}$ is a unit vector pointing in the direction of the outward normal to S. Since (v) involves only the tangential component of $\underline{E}_{h0}$ we have, therefore,

$$\mho I = -\iint_{\text{over } s} \underline{E}_{c0} \cdot \underline{dJ}$$

or

$$I^2 Z = \mho^2 Y = -\iint_S \underline{E} \cdot \underline{J}\, dS$$

where $\underline{E}$ is the electric field due to $\underline{J}$ in the absence of the perfectly conducting parts of the antenna, and $\underline{J}$ is the current distribution on the conductor formed when these parts are shorted at the input terminals and excited as illustrated in Fig. 4.

When applied to the ferrite loop antenna we take S as the surface of the shorted loop. The presence of the ferrite core makes no difference to the argument.

The part of the analysis which begins with Eqs. (i) and (ii) can alternatively be expressed more elegantly in terms of the reaction concept as follows. From (ii) we have

$$\mho I = \langle hh \rangle_1 = \langle hh \rangle_0 + \langle hc \rangle_0$$

$$\langle hh \rangle_0 \to 0 \text{ as } l \to 0.$$

$\langle hc \rangle_0 = -\langle cc \rangle_0$ from the boundary condition at a perfect conductor.

$$\therefore I^2 Z = \mho^2 Y = \mho I = -\langle cc \rangle_0.$$

APPENDIX II

DERIVATION OF EQUATION 11

As was stated earlier, one can write several different formulas for the reaction, < wf >. Different approximations for < wf > are obtained by replacing $\underline{f}$ by a reasonable assumption. The optimum approximation is obtained by adjusting the level of the assumed distribution, a_p, to make the different approximations give the same result. This procedure leads to Eq. 11 in the text. Explicitly, the reaction between w, the electric currents in the windings, and f, the magnetic dipoles in the ferrite, can be expressed in any of the following four forms:

$$\iint_{\text{over } w} \underline{J}_w \cdot \underline{E}_f \, dS \text{ by definition} \tag{i}$$

$$-\iiint_{\text{over } f} \underline{K}_f \cdot H_w dV \text{ by definition} \tag{ii}$$

$$-\iiint_{\text{over core}} j\omega(\mu-\mu_o)\,(\underline{H}_f + \underline{H}_w) \cdot \underline{H}_w dV \text{ which follows from} \tag{iii}$$

$$\underline{K}_f = j\omega(\mu-\mu_o)\,(\underline{H}_w + \underline{H}_f)$$

$$-\iiint_{\text{over core}} \underline{K}_f \cdot \left(\frac{\underline{K}_f}{j\omega(\mu-\mu_o)} - \underline{H}_f\right) dV \text{ which follows from the same relation.} \tag{iv}$$

Note that each form is expressed in terms of two vectors:- (i) $\underline{J}_w$, $\underline{E}_f$; (ii) $\underline{K}_f$, $\underline{H}_w$; (iii) $\underline{H}_f$, $\underline{H}_w$; (iv) $\underline{K}_f$, $\underline{H}_f$. All of these forms are equal as they stand but if we replace the correct source, f, by its approximation, a_p, they are no longer all the same. The reciprocity theorem ensures that (i) and (ii) are still the same, but that is all. We thus obtain three different approximations for the reaction < wf > by replacing f with a_p. The optimum approximation would thus be obtained if we could so choose a_p to make all three the same. Considering that we have only one degree of adjustment at our disposal (the level of a_p), the best we can do is to make two of them the same. The decision as to which two should be made the same can be settled by inspection of (i), (ii), (iii), (iv). Note that $\underline{J}_w$ is known---indeed we assume it to be a constant line density I---

but $\underline{H}_w$, the magnetic field of the loop in free space, is not known explicitly. If we start from an assumed $\underline{K}_f$ then in principle $\underline{H}_f$ and $\underline{E}_f$ are not known explicitly either, but if we take $\underline{K}_f$ as the magnetostatic distribution corresponding to the problem of the core in a uniform field, and if the core is ellipsoidal, then $\underline{H}_f$ and $\underline{E}_f$ can be calculated very much more simply than $\underline{H}_w$. Thus, as a practical matter, we are forced to reject (ii) and (iii) and hence the optimum approximation is given by equating (i) and (iv) which also agrees with (ii) because of the reciprocity theorem. Replacing f by a in (i) and (iv) then gives Eqs. 11 and 12.

A further point in favor of (iv) rather than (iii) is that (iv) gives a stationary formula for the impedance whereas (iii) does not. This is demonstrated in Appendix III.

APPENDIX III

STATIONARY PROPERTIES

To demonstrate the stationary properties of Eq. 11, let δa represent a variation of the approximate source, a_p. Since the approximate impedance is given by Eq. 10 with a in place of f: -

$$I^2 (Z - Z_o) = < w, a > \tag{10}$$

We have

$$I^2 \delta Z = < w, \delta a >$$

where δZ represents the variation in Z due to the variation in a. Now $< w, \delta a >$ is given by Eq. 11 in the form

$$< w, \delta a > = - < a, \delta a > - < \delta a \ a > - < \delta a \ \delta a >$$

$$- \iiint_{core} \frac{2\delta \underline{K}_a \cdot \underline{K}_a}{j\omega (\mu - \mu_o)} dV - \iiint \frac{\delta \underline{K}_a \cdot \delta \underline{K}_a \, dV}{j\omega (\mu - \mu_o)}$$

(which follows directly from the fact that both a and $a + \delta a$ must satisfy Eq. 11.) By the reciprocity theorem

$$< a, \delta a > = < \delta a \ a >$$

Also, for a variation about the correct solution, we can substitute the correct source f for a in the formula for the variation. Thus we can replace

$$\frac{K_a}{j\omega (\mu - \mu_o)}$$

by

$$\frac{K_f}{j\omega(\mu - \mu_o)} = \underline{H}_f + \underline{H}_w$$

$$\therefore \iiint_{core} \frac{2\delta \underline{K}_a \cdot \underline{K}_a \, dV}{j\omega(\mu - \mu_o)} = \iiint 2\delta K_a \cdot (\underline{H}_f + \underline{H}_w) \, dV = -2<\delta a, f> -2<\delta a, w>.$$

Substitution of these relations in the formula for $< w, \delta a >$ gives

$$< w, \delta a > = -2< \delta a, f > - < \delta a, \delta a> + 2< \delta a, f> + 2< \delta a, w >$$

$$< w, \delta a > = < \delta a, \delta a>.$$

Thus the variation of Z is of the second order for variations of a about the correct distribution.

On the other hand, if we used (i) and (iii) of Appendix II to obtain an approximate formula for Z, the formula for a would take the form (on replacing f by a)

$$< w\ a > = - \iiint_{core} j\omega(\mu - \mu_o)\ (\underline{H}_a + \underline{H}_w) \cdot H_w\ dV.$$

Now

$$< w\ \delta a> = - \iiint_{core} j\omega(\mu - \mu_o)\ \delta\ \underline{H}_a \cdot \underline{H}_w\ dV$$

which is of the first order, since $\delta\ \underline{H}_a$ is an arbitrary variation, independent of $\underline{H}_w$. The formula so obtained therefore is not stationary.

APPENDIX IV

CALCULATION OF INDUCTANCE USING STATIC THEORY

According to static theory, a winding of n turns on the surface of an ellipsoid uniformly spaced along an axis gives a uniform internal field as $n \to \infty$. In the limit the winding can be represented by a surface density, $\underline{J}$. It is found that $\underline{J}$ is related to the internal field, $\underline{H}$, by the equation

$$\underline{J} = \underline{H} \times \underline{N} \, \frac{1 + \mathcal{D}(k-1)}{1 - \mathcal{D}} \tag{i}$$

where

$\mathcal{D}$ = demagnetization factor

N = unit vector in direction of outward normal.

To find the inductance, we can start with the formula for the input voltage, $\mathcal{V}$:

$$\mathcal{V} = -\oint \underline{E} \cdot \underline{dl}$$

the integral being taken along the wire and across the gap. Thus

$$\mathcal{V} = \sum_{\text{all turns}} \; -\int_{\text{round each turn}} \underline{E} \cdot \underline{dl}$$

$$= \sum_{\text{all turns}} \; -\iint_{\text{over each cross section}} \underline{\nabla} \times \underline{E} \cdot \underline{\hat{z}} \, dS \text{ as } n \to \infty$$

where $\underline{\hat{z}}$ represents a unit vector in the direction of the axis

$$= \sum \iint_{\text{over each cross section}} j\omega\mu \, \underline{H} \cdot \underline{\hat{z}} \, dS$$

$$= \frac{1}{s} \iiint_{\text{volume of core}} j\omega\mu \, \underline{H} \cdot \underline{\hat{z}} \, dV$$

$$= \frac{1}{s} \iiint_{\text{volume of core}} j\omega\mu \, H \, dV \text{ since } \underline{H} \text{ is parallel to } \underline{\hat{z}}. \tag{ii}$$

where

s = spacing between turns, measured along the axis

$$s = \frac{2a}{n}$$

where

$$2a = \text{length of axis.}$$

Now to express the input current, I, in terms of H, note that by definition

$$s\, \underline{J} = \underline{\hat{z}} \times \underline{N}\, I,$$

Comparison with Eq. (i) shows that

$$I = sH \frac{[1+\mathcal{D}(k-1)]}{1-\mathcal{D}}. \tag{iii}$$

Since H is constant (ii) gives

$$\mathcal{U} = \frac{1}{s}\, j\omega\mu\, HV \tag{iv}$$

where V = volume of core.

$$L = \frac{\mathcal{U}}{j\omega I} = \frac{\mu V\, (1-\mathcal{D})}{s^2(1+\mathcal{D}[k-1])}. \tag{v}$$

It can easily be verified that this is consistent with Eq. 20.

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